Extra slides

March 16, 2016

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Recall slides 30 and 35 in Lecture 1.

What is $\mathbb{G}_\mathcal{X}$?

Slide 30

Suppose $\mathcal{X}$ is a nonempty set. A *gamble* on $\mathcal{X}$ is an extended real-valued function on $\mathcal{X}$ that is bounded below.

Call a set $\mathbf{G}$ of gambles on $\mathcal{X}$ an *offer* on $\mathcal{X}$ if it obeys these five axioms:

Axiom G0. If $g - \epsilon \in \mathbf{G}$ for all $\epsilon > 0$, then $g \in \mathbf{G}$.

Axiom G1. If $g_1, g_2 \in \mathbf{G}$, then $g_1 + g_2 \in \mathbf{G}$.

Axiom G2. If $c \in [0, \infty)$ and $g \in \mathbf{G}$, then $cg \in \mathbf{G}$.

Axiom G3. If $g_1 \in \mathbf{G}$, $g_2 \in \mathbb{G}_{\mathcal{X}}$, and $g_2 \leq g_1$, then $g_2 \in \mathbf{G}$.

Axiom G4. If $g \in \mathbf{G}$, then $\inf g \leq 0$.

Slide 35

Skeptic announces $\mathcal{K}_0 \in \mathbb{R}$.
FOR $n = 1, 2, \ldots$:
Forecaster announces an upper expectation $\overline{\mathbf{E}}_n$ on $\mathcal{X}$.
Skeptic announces $f_n \in \mathbb{G}_{\mathcal{X}}$ such that $\overline{\mathbf{E}}_n(f_n) \leq \mathcal{K}_{n-1}$.
Reality announces $x_n \in \mathcal{X}$.
$\mathcal{K}_n := f_n(x_n)$.

Classical probability theory constructs global probability measures (stochastic processes) from local probability measures (conditional probabilities).

In game-theoretic probability, the construction of global expected values uses Skeptic's strategies.

The upper expected value $\overline{\mathbb{E}}(X)$ for a global variable $X(\overline{\mathbf{E}}_1, x_1, \overline{\mathbf{E}}_2, x_2, \ldots)$ is the least initial stake Skeptic needs to obtain X at the end of the game.

Intuitively, $\mathbb{G}_{\mathcal{X}}$ is the set of all gambles on Reality's move x.

A gamble on x is a real-valued function on $\mathcal{X}$.

But should we require a gamble to be bounded? Or, on the contrary, should we consider unbounded functions and even extended real-valued functions—ones that can take the values ∞ and $-\infty$?

Various authors answer in various ways at various times.

- Peter Walley, the initiator of imprecise probabilities, tended to consider only bounded functions.
- In the 2011 Shafer/Vovk/Takemura paper on Lévy's zero-one law, we went to the other extreme, allowing gambles to be unbounded and take the values ∞ and $-\infty$.
- I now think the mathematics is simplest if we allow a gamble to take the value ∞ but require it to be bounded below.

Lévy's zero-one law in game-theoretic probability (with Vladimir Vovk and Akimichi Takemura). *Journal of Theoretical Probability* 25(1):1-24. DOI: 10.1007/s10959-011-0390-3. 2012.

Slide 35

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$\overline{\mathbf{E}}_n$ is an upper expectation on $\mathcal{X}$, whereas $\overline{\mathbb{E}}$ is an upper expectation on Ω, the set of all paths $\overline{\mathbf{E}}_1, x_1, \overline{\mathbf{E}}_2, x_2, \ldots$.

Two-player game

Skeptic announces $\mathcal{K}_0 \in \mathbb{R}$.
FOR $n = 1, 2, \ldots, N$:
  Skeptic announces $L_n \in \mathbb{R}$.
  Reality announces $y_n \in \{0, 1\}$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + L_n(y_n - \frac{1}{2})$.

An *event* E is a property of Reality's moves.
$\mathbb{P}(E) :=$ stake Skeptic needs to obtain 1 if E happens, 0 otherwise.
Example: $E := \{y_1 = y_2 = 1\}$. By Pascal's argument, $\mathbb{P}(E) = 1/4$.

A *variable* X is a function of Reality's moves.
$\mathbb{E}(X) :=$ stake Skeptic needs to obtain $X(y_1, y_2, \ldots)$.
Example: $X(y_1, y_2, \ldots) := y_1 + y_2$. Then $\mathbb{E}(X) = 1$.

Three-player game

Skeptic announces $\mathcal{K}_0 \in \mathbb{R}$.
FOR $n = 1, 2, \ldots$:
  Forecaster announces $p_n \in [0, 1]$.
  Skeptic announces $L_n \in \mathbb{R}$.
  Reality announces $y_n \in \{0, 1\}$.
  $\mathcal{K}_n := \mathcal{K}_{n-1} + L_n(y_n - p_n)$.

An *event* E is a property of Forecaster's and Reality's moves.
$\overline{\mathbb{P}}(E) :=$ stake Skeptic needs to obtain at least 1 if E happens, 0 otherwise.
$\underline{\mathbb{P}}(E) := 1 - \overline{\mathbb{P}}(E^c)$.
Example: $E := \{p_1 < \frac{1}{2} \,\&\, y_1 = 1\}$. Then $\overline{\mathbb{P}}(E) = \frac{1}{2}$ and $\underline{\mathbb{P}}(E) = 0$.

A *variable* X is a function of Forecaster's and Reality's moves.
$\overline{\mathbb{E}}(X) :=$ stake Skeptic needs to obtain at least $X(p_1, y_1, p_2, y_2, \ldots)$.
$\underline{\mathbb{E}}(X) := -\overline{\mathbb{E}}(-X)$.
Example: $X(p_1, y_1, p_2, y_2, \ldots) := y_1 - p_1$. Then $\overline{\mathbb{E}}(X) = \underline{\mathbb{E}}(X) = 0$.

Protocol 1.3. Testing forecasts of zero for bounded outcomes

Protocol:

Skeptic announces $\mathcal{K}_0 \in \mathbb{R}$.

FOR $n = 1, 2, \ldots$:

Skeptic announces $M_n \in \mathbb{R}$.

Reality announces $x_n \in [-1, 1]$.

$\mathcal{K}_n := \mathcal{K}_{n-1} + M_n x_n$.

Skeptic begins with $\mathcal{K}_0 = 1$ and always sets $M_n = \kappa \mathcal{K}_{n-1}$, where $0 < \kappa < 1/2$. As one can easily see, this produces the martingale $\mathcal{M}$ given by

$$\mathcal{M}(x_1 \ldots x_n) := \prod_{i=1}^{n} (1 + \kappa x_i) \,. \tag{1.8}$$

$$\mathcal{M}(x_1 \dots x_n) = \mathcal{M}(x_1 \dots x_{n-1})(1 + \kappa x_n) = \prod_{i=1}^{n} (1 + \kappa x_i) \, .$$

Consider the process $\mathcal{T}$ defined by $\mathcal{T}(\square) = 1$ and

$$\mathcal{T}(x_1 \dots x_n) := \prod_{i=1}^{n} \exp\left(\kappa x_i - \kappa^2 x_i^2\right) = \exp\left(\kappa \sum_{i=1}^{n} x_i - \kappa^2 \sum_{i=1}^{n} x_i^2\right) . \quad (1.23)$$

For $t \geq -1/2$, $\ln(1+t) \geq t - t^2$, and therefore $1+t \geq \exp(t-t^2)$. So $\mathcal{T}$ is multiplied by a smaller positive factor on each round than $\mathcal{M}$ and is therefore bounded above by $\mathcal{M}$. Let $\omega = x_1 x_2 \dots$ be a path such that $\sup_n \mathcal{M}(x_1 \dots x_n) < \infty$. Then there exists a constant $C_\omega > 0$ such that $\mathcal{M}(x_1 \dots x_n) \leq C_\omega$ and hence $\mathcal{T}(x_1 \dots x_n) \leq C_\omega$ for all n. This implies that ω satisfies

$$\kappa \sum^{n} x_i - \kappa^2 \sum^{n} x_i^2 \leq \ln C_\omega,$$

or

$$\kappa \sum_{i=1}^{n} x_i - \kappa^2 n \leq \ln C_\omega,$$

or

$$\overline{x}_n \leq \frac{\ln C_\omega}{\kappa n} + \kappa$$

American Statistical Association
Statement on p-values, March 2016.

Summary

1. P-values can indicate how incompatible the data are with a specified statistical model.
2. P-values do not measure the probability that the studied hypothesis is true, or the probability that the data were produced by random chance alone.
3. Scientific conclusions and business or policy decisions should not be based only on whether a p-value passes a specific threshold.
4. Proper inference requires full reporting and transparency.
5. A p-value, or statistical significance, does not measure the size of an effect or the importance of a result.
6. By itself, a p-value does not provide a good measure of evidence regarding a model or hypothesis

This is a very mild critique of p-values, designed to earn the assent of the entire statistical profession.